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- Mathematical **theory** dealing with **aggregation** of preferences.
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Dimitris Fotakis  
Approximate Mechanism Design without Money
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- Axiomatic framework and impossibility result by Arrow (1951).
- Collective decision making, by **voting**, over anything:
  - Political representatives, award nominees, contest winners, allocation of tasks/resources, joint plans, meetings, food, …
  - Web-page ranking, preferences in multiagent systems.

Formal Setting

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## Colors of the Local Football Club?

Preferences of the founders about the colors of the local club:

- **12 boys**: Green $\succ$ Red $\succ$ Pink
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An Example

Colors of the Local Football Club?

Preferences of the founders about the colors of the local club:

- 12 boys: Green ≻ Red ≻ Pink
- 10 boys: Red ≻ Green ≻ Pink
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With plurality voting \((1, 0, 0)\): Green\((12) \succ Red\((10) \succ Pink\((3)\)
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Probably it would have been Red$(13) \succ$ Green$(12) \succ$ Pink$(0)$
## Positional Scoring Voting Rules

- Vector \((a_1, \ldots, a_m)\), \(a_1 \geq \cdots \geq a_m \geq 0\), of **points** allocated to each **position** in the preference list.

- **Winner** is the alternative getting **most points**.
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Borda Count (1770): \((m - 1, m - 2, \ldots, 1, 0)\)

“Intended only for honest men.”
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- (Green, Red): (12, 13), (Green, Pink): (22, 3), (Red, Pink): (22, 3)

---

**Condorcet paradox**: Condorcet winner may not exist.

- $a \succ b \succ c$, $b \succ c \succ a$, $c \succ a \succ b$ ($a, b$):
  1. (2, 1)
  2. (a, c):
  3. (b, c):

**Condorcet criterion**: select the Condorcet winner, if exists.

- Plurality satisfies the Condorcet criterion?
- Borda count?

**“Approximation” of the Condorcet winner**: Dodgson (NP-hard to approximate!), Copeland, MiniMax, ...
**Condorcet Winner**

- **Winner** is the alternative **beating every other** alternative in pairwise election.
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### Social Choice

#### Setting
- Set $A$ of possible **alternatives** (candidates).
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Desirable Properties of Social Choice Functions

- **Onto**: Range is $A$.
- **Unanimous**: If $a$ is the top alternative in all $\succ_1, \ldots, \succ_n$, then
  \[ F(\succ_1, \ldots, \succ_n) = a \]
- **Not dictatorial**: For each agent $i$, $\exists \succ_1, \ldots, \succ_n$:
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  $F(\succ_1, \ldots, \succ_n) \neq$ agent’s $i$ top alternative
- **Strategyproof or truthful**: $\forall \succ_1, \ldots, \succ_n, \forall$ agent $i$, $\forall \succ'_i$,
  
  $F(\succ_1, \ldots, \succ_i, \ldots, \succ_n) \succ_i F(\succ_1, \ldots, \succ'_i, \ldots, \succ_n)$
Gibbard-Satterthwaite Theorem (mid 70’s)

Any strategyproof and onto social choice function on more than 2 alternatives is \textit{dictatorial}.
Gibbard-Satterthwaite Theorem (mid 70’s)

Any \textit{strategyproof} and \textit{onto} social choice function on \textit{more than 2} alternatives is \textit{dictatorial}.

Escape Routes

- Randomization
- Monetary payments
- Voting systems \textit{computationally hard} to manipulate.
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Escape Routes

- Randomization
- Monetary payments
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- Restricted domain of preferences – Approximation
Single Peaked Preferences

- One dimensional ordering of alternatives, e.g. \( A = [0, 1] \)
- Each agent \( i \) has a **single peak** \( x_i^* \in A \) such that for all \( a, b \in A \):
  \[
  b < a \leq x_i^* \quad \Rightarrow \quad a \succ_i b \\
  x_i^* \geq a > b \quad \Rightarrow \quad a \succ_i b
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Single Peaked Preferences and Medians

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Median Voter Scheme [Moulin 80], [Sprum 91], [Barb Jackson 94]

A social choice function $F$ on a single peaked preference domain is **strategyproof**, **onto**, and **anonymous** iff there exist $y_1, \ldots, y_{n-1} \in A$ such that for all $(x_1^*, \ldots, x_n^*)$,

$$F(x_1^*, \ldots, x_n^*) = \text{median}(x_1^*, \ldots, x_n^*, y_1, \ldots, y_{n-1})$$
Strategic Agents in a Metric Space

- Set of agents $N = \{1, \ldots, n\}$
- Each agent $i$ wants a facility at $x_i$. Location $x_i$ is agent $i$’s private information.
Strategic Agents in a Metric Space

- Set of agents $N = \{1, \ldots, n\}$
- Each agent $i$ wants a facility at $x_i$. Location $x_i$ is agent $i$’s private information.
- Each agent $i$ reports that she wants a facility at $y_i$. Location $y_i$ may be different from $x_i$. 

![Diagram showing three agents with different wants and reports](image-url)
(Randomized) Mechanism

A social choice function $F$ that maps a location profile $y = (y_1, \ldots, y_n)$ to a (probability distribution over) set(s) of $k$ facilities.
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Connection Cost

(Expected) distance of agent $i$’s true location to the nearest facility:

$$\text{cost}[x_i, F(y)] = d(x_i, F(y))$$

[Diagram showing connection cost with distances $a < b < c$.]
Desirable Properties of Mechanisms

**Strategyproofness**

For any location profile $x$, agent $i$, and location $y$:

$$\text{cost}[x_i, F(x)] \leq \text{cost}[x_i, F(y, x_{-i})]$$
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**Efficiency**

$F(x)$ should optimize (or approximate) a given objective function.

- **Social Cost**: minimize $\sum_{i=1}^{n} \text{cost}[x_i, F(x)]$
- **Maximum Cost**: minimize $\max\{\text{cost}[x_i, F(x)]\}$
Desirable Properties of Mechanisms

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- Minimize $p$-norm of $(\text{cost}[x_1, F(x)], \ldots, \text{cost}[x_n, F(x)])$
The median of $(x_1, \ldots, x_n)$ is strategyproof and optimal.
The median of \((x_1, \ldots, x_n)\) is strategyproof and optimal.
1-Facility Location on the Line

The **median** of \((x_1, \ldots, x_n)\) is **strategyproof** and **optimal**.
1-Facility Location in Other Metrics

1-Facility Location in a Tree [Schummer Vohra 02]

- **Extended medians** are the **only** strategyproof mechanisms.
- **Optimal** is an extended median, and thus **strategyproof**.
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1-Facility Location in General Metrics

- Any onto and strategyproof mechanism is a dictatorship [SV02]
- The optimal solution is not strategyproof!
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- Any **onto** and **strategyproof** mechanism is a **dictatorship** [SV02]
- The optimal solution is **not strategyproof**!
- Deterministic **dictatorship** has cost $\leq (n - 1)\text{OPT}$.
- Randomized **dictatorship** has cost $\leq 2\text{OPT}$ [Alon FPT 10]
The optimal solution is **not strategyproof**!
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2-Facility Location on the Line

The optimal solution is not strategyproof!

\[ y_1 = -1 - 2\varepsilon \quad x_2 = 0 \quad x_3 = 1 + \varepsilon \]
The optimal solution is **not strategyproof**!

**Two Extremes Mechanism** [Procacc Tennen 09]

- Facilities at the **leftmost** and at the **rightmost** location:
  \[ F(x_1, \ldots, x_n) = (\min\{x_1, \ldots, x_n\}, \max\{x_1, \ldots, x_n\}) \]
- **Strategyproof** and \((n - 2)\)-approximate.
Approximate Mechanism Design Design [Procacc Tennen 09]

- Sacrifice **optimality** for **strategyproofness**.
- **Best approximation** ratio by **strategyproof** mechanisms?
- Variants of $k$-Facility Location, $k = 1, 2, \ldots$, among the **central** problems in this research agenda.
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2-Facility Location on the Line – Approximation Ratio

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Deterministic 2-Facility Location on the Line

**Nice** mechanisms ≡ deterministic **strategyproof** mechanisms with a **bounded approximation** (function of $n$ and $k$).

Niceness **objective-independent** and **facilitates** the characterization!
Deterministic 2-Facility Location on the Line

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Niceness **objective-independent** and **facilitates** the characterization!

Any **nice** mechanism $F$ for $n \geq 5$ agents:

- Either $F(x) = (\min x, \max x)$ for all $x$ (Two Extremes).
- Or admits unique **dictator** $j$, i.e., $x_j \in F(x)$ for all $x$. 

---

**Dictatorial Mechanism with Dictator $j$**

Consider distances $d_l = x_j - \min x$ and $d_r = \max x - x_j$.

Place the first facility at $x_j$ and the second at $x_j - \max\{d_l, 2d_r\}$, if $d_l > d_r$, and at $x_j + \max\{2d_l, d_r\}$, otherwise.

**Strategyproof** and $(n - 1)$-approximate.
Deterministic 2-Facility Location on the Line

Nice mechanisms ≡ deterministic strategyproof mechanisms with a bounded approximation (function of \( n \) and \( k \)).

Niceness objective-independent and facilitates the characterization!

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Dictatorial Mechanism with Dictator \( j \)

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- Strategyproof and \((n - 1)\)-approximate.
Two Extremes is the only anonymous nice mechanism for allocating 2 facilities to \( n \geq 5 \) agents on the line.

The approximation ratio for 2-Facility Location on the line by deterministic strategyproof mechanisms is \( n - 2 \).
**Consequences**

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**Deterministic \( k \)-Facility Location, for all \( k \geq 3 \)**

There are **no anonymous nice** mechanisms for \( k \)-Facility Location for all \( k \geq 3 \) (even on the line and for \( n = k + 1 \)).
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There are **no anonymous nice** mechanisms for \( k \)-Facility Location for all \( k \geq 3 \) (even on the line and for \( n = k + 1 \)).

Deterministic 2-Facility Location in General Metrics

There are **no nice** mechanisms for 2-Facility Location in metrics more general than the line and the cycle (even for 3 agents in a **star**).
Proportional Mechanism

Facilities open at the locations of selected agents.

1st Round: Agent $i$ is selected with probability $1/n$

2nd Round: Agent $j$ is selected with probability $\frac{d(x_j, x_i)}{\sum_{\ell \in N} d(x_{\ell}, x_i)}$
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Strategyproof and $4$-approximate for general metrics.

Not strategyproof for >2 facilities!

Profile $(0: \text{many}, 1: 50, 1: 50, 5: 4, 101: 101, 5: 1, 1: 1 + 105, 5: 1, 1 + 105)$. 

Dimitris Fotakis

Approximate Mechanism Design without Money
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- Strategyproof and 4-approximate for general metrics.
Randomized 2-Facility Location

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- **Strategyproof** and 4-approximate for general metrics.
- **Not strategyproof** for $> 2$ facilities!
  Profile $(0: \text{many}, 1: 50, 1 + 10^5: 4, 101 + 10^5: 1), 1 \to 1 + 10^5$. 

Dimitris Fotakis
Approximate Mechanism Design without Money
Randomized $k$-Facility Location for $k \geq 3$ [F. Tzamos 10]

**Winner-Imposing Mechanisms**

- Agents with a facility at their reported location connect to it. Otherwise, no restriction whatsoever.

---

connection cost = $a$ \hspace{1cm} (a < b < c)
Winner-Imposing Mechanisms

- Agents with a facility at their reported location connect to it. Otherwise, no restriction whatsoever.
- Winner-imposing version of the Proportional Mechanism is strategyproof and $4k$-approximate in general metrics, for any $k$. 

\[
\begin{align*}
\text{connection cost} &= a \\
&\quad (a < b < c) \\
\end{align*}
\]

\[
\begin{align*}
\text{connection cost} &= c \\
&\quad (a < b < c) \\
\end{align*}
\]
Equal-Cost Mechanism

- **Optimal maximum** cost $\text{OPT} = C/2$.
- **Cover** all agents with $k$ disjoint intervals of length $C$.

Diagram:

- Agents located at $x_1, x_2, x_3, x_4, \ldots, x_i, \ldots, x_{n-1}, x_n$ with length $C$.
Equal-Cost Mechanism

- **Optimal maximum** cost $\text{OPT} = C/2$.
- **Cover** all agents with $k$ disjoint intervals of length $C$.
- Place a facility to an **end** of each interval.
  
  With prob. $1/2$, facility at $L - R - L - R - \ldots$
  
  With prob. $1/2$, facility at $R - L - R - L - \ldots$

**Agents' Cost and Approximation Ratio**

Agent $i$ has expected cost $\frac{C - x_i}{2} + \frac{x_i}{2} = \frac{C}{2} = \text{OPT}$.

Approx. ratio: 2 for the maximum cost, $n$ for the social cost.
Randomized $k$-Facility Location on the Line

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Agents’ Cost and Approximation Ratio

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---

$x_1 \ x_2 \ x_3 \ x_4 \ \ldots \ \ldots \ x_i \ \ldots \ x_{n-1} \ x_n$
**Randomized $k$-Facility Location on the Line**

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- Agent $i$ has expected cost $\text{cost} = (C - x_i)/2 + x_i/2 = C/2 = \text{OPT}$.
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Randomized $k$-Facility Location on the Line

Equal-Cost Mechanism

- **Cover** all agents with $k$ disjoint intervals of length $C$.
- Place a facility to an end of each interval.

Strategyproofness

- Agents do not have incentives to lie and increase OPT.
- Let agent $i$ declare $y_i$ and decrease OPT to $C'/2 < C/2$.

![Diagram of equal-cost mechanism with intervals and facilities](image-url)
Equal-Cost Mechanism

- **Cover** all agents with *k disjoint intervals* of length \( C \).
- Place a facility to an **end** of each interval.

Strategyproofness

- Agents do not have incentives to lie and increase OPT.
- Let agent \( i \) declare \( y_i \) and decrease OPT to \( C'/2 < C/2 \).
- Distance of \( x_i \) to nearest \( C' \)-interval \( \geq C - C' \).
**Equal-Cost Mechanism**

- **Cover** all agents with \( k \) disjoint intervals of length \( C \).
- Place a facility to an **end** of each interval.

**Strategyproofness**

- Agents do not have incentives to lie and increase OPT.
- Let agent \( i \) declare \( y_i \) and **decrease** OPT to \( C'/2 < C/2 \).
- Distance of \( x_i \) to **nearest** \( C'\)-interval \( \geq C - C' \).
- \( i \)'s expected cost \( \geq (C - C')/2 + C/2 = C - C'/2 > C/2 \)
Randomized $k$-Facility Location on the Line [F. Tzamos 13]

Equal-Cost Mechanism

- **Cover** all agents with $k$ disjoint intervals of length $C$.
- Place a facility to an **end** of each interval.

Agents with Concave Costs

**Generalized** Equal-Cost Mechanism is **strategyproof** and has the **same approximation** ratio if agents’ cost is a **concave function** of distance to the nearest facility.
Understanding the Power of Verification

- (Implicit or explicit) **verification** restricts agents’ declarations.
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  - **ε-verification**: agent $i$ at $x_i$ can **only** declare anything in $[x_i - \varepsilon, x_i + \varepsilon]$.
  - **Winner-imposing**: lies that increase mechanism’s cost cause a (proportional) **penalty** to the agent [F. Tzamos 10] [Koutsoupias 11]
Understanding the Power of Verification

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  - **$\varepsilon$-verification** : agent $i$ at $x_i$ can **only** declare anything in $[x_i - \varepsilon, x_i + \varepsilon]$, [Carag. Elk. Szeg. Yu 12] [Archer Klein. 08]
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- **Non-symmetric** verification: **conditions** under which the mechanism gets some **advantage**.
Research Directions

Understanding the Power of Verification

- (Implicit or explicit) verification restricts agents’ declarations.
  - \(\varepsilon\)-verification: agent \(i\) at \(x_i\) can only declare anything in \([x_i - \varepsilon, x_i + \varepsilon]\), [Carag. Elk. Szeg. Yu 12] [Archer Klein. 08]
  - Winner-imposing: lies that increase mechanism’s cost cause a (proportional) penalty to the agent [F. Tzamos 10] [Koutsoupias 11]

- Non-symmetric verification: conditions under which the mechanism gets some advantage.

Voting and Social Networks

- How group of people vote for their leader in social networks?
- How social network affects the people’s votes and the outcome? Relation to opinion dynamics?
Thank You!